Int. J. Heat Mass Transfer. Vol. 36, No. 18, pp. 4467–4475, 1993
Printed in Great Britain

0017-9310/93 $6.00+0.00

Integral transform solution of Luikov's equations for heat and mass transfer in capillary porous media

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(*Received* 30 *March* 1993 *and in final form* 29 *April* 1993)

Abstract—The Luikov system of equations for coupled heat and mass transfer within capillary porous bodies is analytically handled through application of the generalized integral transform technique. The problem of temperature and moisture distribution during contact drying of a moist porous sheet is considered to illustrate the development of the present approach. The classical coupled auxiliary problem with the related complex eigenvalues is completely avoided and, instead, two decoupled eigenvalue problems for temperature and moisture are chosen, which are of the conventional Sturm–Liouville type. A set of benchmark results is generated and critically compared with previously reported approximate solutions.

INTRODUCTION

THE SO-CALLED Luikov's equations provide a well-established model for the analysis of various simultaneous heat and mass diffusion problems in capillary porous media, and have been reviewed in different sources [1–4]. Pertinent applications include the drying of wood, ceramics and bricks, moisture migration in soils, and the analysis of heat pipe wicks. In the case of a linear formulation for constant transport coefficients, analytical solutions were proposed over the years, based on both the Laplace transform method [2, 5] and the classical integral transform method [6–8]. It was later observed that the numerical results obtained through such analysis could be in error, due to the existence of complex eigenvalues in the associated coupled auxiliary problem [9, 10]. Then, quite recently, the effects of including one pair of complex conjugate eigenvalues in the analytical infinite summations was critically investigated [10, 11]. Specially for shorter times, the need for considering the complex eigenvalues is crucial, since even the qualitative behavior of the solution may be erroneously predicted. Therefore, the present paper brings an alternative analytical solution to this class of problems, by completely avoiding the difficulties associated with the computation of complex eigenvalues, and yielding some freedom in selection of the basis for the eigenfunction expansions. To achieve this goal the ideas in the generalized integral transform technique [12–21] are further extended, by gathering information on the solution of coupled problems [20, 21] and nonhomogeneous diffusion problems [22]. The proposed approach is not limited to linear situations, following the flexibility introduced through the generalized integral transform technique [12–14], which provides hybrid numerical–analytical solutions to nonlinear problems. In order to illustrate this approach, an application on contact drying of a porous moist sheet is considered more closely, for the case of general third kind boundary conditions and constant physical properties. A pair of independent auxiliary problems for the temperature and moisture eigenfunction expansions is chosen, which are of the conventional Sturm–Liouville type and, therefore, involve real quantities only. An infinite system of coupled ordinary differential equations for the transformed potentials then results, upon integral transformation of the original partial differential equations. A sufficiently large finite system, for the required accuracy, is then obtained by truncation of the denumerable system at the Nth row and column, which can be either analytically or numerically handled through widely available scientific subroutines libraries [23]. The convergence behavior of the proposed eigenfunction expansions is here illustrated through representative numerical examples and a set of benchmark results for reference purposes is produced. Previous analytical solutions are critically examined, either without any complex eigenvalues [7] or including one pair of conjugate complex roots [10].

ANALYSIS

The proposed approach is here applied to a typical heat and mass transfer problem governed by Luikov's equations, related to contact drying of a porous moist sheet on a hot plate, as depicted in Fig. 1. This drying problem is also solved in refs. [7, 8] without con-

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NOMENCLATURE

a	thermal diffusivity of the porous medium
a_m	diffusion coefficient of moisture in the porous medium
c	specific heat of porous medium
h	heat transfer coefficient
h_m	mass transfer coefficient
k	thermal conductivity
k_m	moisture conductivity
l	thickness of porous sheet
N	order of truncated system
q	prescribed heat flux
r	latent heat of evaporation
t	time
$T(x, t)$	temperature distribution
T_s	temperature of surrounding air
T_0	initial temperature distribution
$u(x, t)$	moisture distribution
u^*	moisture in equilibrium with surrounding air
u_0	initial moisture distribution
x	position.

Greek symbols

γ_n	eigenvalues of matrix **C**, from problem (24b)
δ	thermogradient coefficient
ε	phase change criterion ($\varepsilon = 0$ all liquid, $\varepsilon = 1$ all vapor)
μ_i, λ_i	eigenvalues of problems (7) and (8), respectively
ψ_i, Γ_i	eigenfunctions of problems (7) and (8), respectively
$\xi^{(n)}$	eigenvectors of matrix **C**, from problem (24b).

Subscripts

i, j	order of eigenquantities from problems (7) and (8)
$k = 1, 2$	related to temperature and moisture, respectively
s	steady-state solution
h	homogeneous problem
av	averaged (integrated) quantity.

sideration of the associated complex eigenvalues, and recently by Lobo *et al.* [10] with the inclusion of one pair of conjugate roots of the transcendental equation. Numerical results from these works were markedly different for the smaller values of dimensionless time, due to the influence of the complex eigenvalues. Therefore, the present alternative solution is demonstrated for this same one-dimensional linear problem, to allow for critical and definitive comparisons. The problem formulation in dimensionless form is given by [7, 8]:

$$\frac{\partial \theta_1(X,\tau)}{\partial \tau} = \frac{\partial^2 \theta_1(X,\tau)}{\partial X^2} - \varepsilon Ko \frac{\partial \theta_2(X,\tau)}{\partial \tau}, \quad \text{in } 0 < X < 1, \quad \tau > 0 \qquad (1a)$$

$$\frac{\partial \theta_2(X,\tau)}{\partial \tau} = Lu \frac{\partial^2 \theta_2(X,\tau)}{\partial X^2} - Lu\,Pn \frac{\partial^2 \theta_1(X,\tau)}{\partial X^2}, \quad \text{in } 0 < X < 1, \quad \tau > 0 \qquad (1b)$$

subject to the initial conditions

$$\theta_1(X,0) = 0, \quad \theta_2(X,0) = 0, \quad \text{in } 0 \leqslant X \leqslant 1 \qquad (1c,d)$$

and boundary conditions given by

$$\frac{\partial \theta_1(0,\tau)}{\partial X} = -Q;$$

$$\frac{\partial \theta_2(0,\tau)}{\partial X} - Pn \frac{\partial \theta_1(0,\tau)}{\partial X} = 0, \quad \tau > 0 \qquad (1e,f)$$

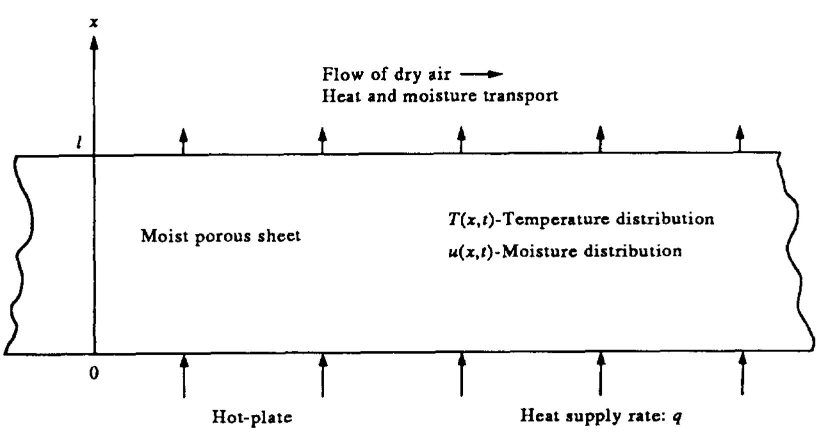

FIG. 1. Geometry and coordinate system for contact drying of a moist porous sheet.

$$\frac{\partial\theta_1(1,\tau)}{\partial X} - Bi_q[1-\theta_1(1,\tau)] + (1-\varepsilon)\,Ko\,Lu\,Bi_m[1-\theta_2(1,\tau)] = 0, \tau > 0 \quad (1g)$$

$$-\frac{\partial\theta_2(1,\tau)}{\partial X} + Pn\frac{\partial\theta_1(1,\tau)}{\partial X} + Bi_m[1-\theta_2(1,\tau)] = 0, \quad \tau > 0 \quad (1h)$$

where various dimensionless groups are defined as

$$\theta_1(X,\tau) = \frac{T(x,t)-T_0}{T_s-T_0}, \quad \text{dimensionless temperature}$$

$$\theta_2(X,\tau) = \frac{u_0-u(x,t)}{u_0-u^*}, \quad \text{dimensionless moisture}$$

$$X = \frac{x}{l}, \quad \text{dimensionless coordinate}$$

$$\tau = \frac{at}{l^2}, \quad \text{dimensionless time}$$

$$Lu = \frac{a_m}{a}, \quad \text{Luikov number}$$

$$Pn = \delta\frac{T_s-T_0}{u_0-u^*}, \quad \text{Possnov number}$$

$$Ko = \frac{r}{c}\cdot\frac{u_0-u^*}{T_s-T_0}, \quad \text{Kossovitch number}$$

$$Bi_q = \frac{hl}{k}, \quad \text{dimensionless heat transfer coefficient}$$

$$Bi_m = \frac{h_m l}{k_m}, \quad \text{dimensionless mass transfer coefficient}$$

$$Q = \frac{ql}{k(T_s-T_0)}, \quad \text{dimensionless heat flux.} \quad (2)$$

Without loss of generality, system (1) is rewritten in a more convenient form as follows:

$$\frac{\partial\theta_1(X,\tau)}{\partial\tau} = \alpha\frac{\partial^2\theta_1}{\partial X^2} - \beta\frac{\partial^2\theta_2}{\partial X^2}, \quad 0 < X < 1, \quad \tau > 0 \quad (3a)$$

$$\frac{\partial\theta_2(X,\tau)}{\partial\tau} = Lu\frac{\partial^2\theta_2}{\partial X^2} - Lu\,Pn\frac{\partial^2\theta_1}{\partial X^2}, \quad 0 < X < 1, \quad \tau > 0 \quad (3b)$$

$$\theta_1(X,0) = \theta_2(X,0) = 0, \quad 0 \leqslant X \leqslant 1 \quad (3c,d)$$

$$\frac{\partial\theta_1(0,\tau)}{\partial X} = -Q, \quad \frac{\partial\theta_2(0,\tau)}{\partial X} = -Pn\,Q, \quad \tau > 0 \quad (3e,f)$$

$$\frac{\partial\theta_1(1,\tau)}{\partial X} + Bi_q\theta_1(1,\tau) = Bi_q - (1-\varepsilon)\,Ko\,Lu\,Bi_m(1-\theta_2(1,\tau)), \quad \tau > 0 \quad (3g)$$

$$\frac{\partial\theta_2(1,\tau)}{\partial X} + Bi_m^*\theta_2(1,\tau) = Bi_m^* - Pn\,Bi_q(\theta_1(1,\tau)-1), \quad \tau > 0 \quad (3h)$$

where,

$$\alpha = 1+\varepsilon\,Ko\,Lu\,Pn; \quad \beta = \varepsilon\,Ko\,Lu; \quad (3i,j)$$

$$Bi_m^* = Bi_m[1-(1-\varepsilon)\,Pn\,Ko\,Lu]. \quad (3k)$$

For best computational performance, the boundary conditions are made homogeneous by separating out the contribution of the steady-state solutions, i.e.

$$\theta_k(X,\tau) = \theta_{ks}(X) + \theta_{kh}(X,\tau), \quad k = 1,2 \quad (4a,b)$$

where the steady-state solutions, $\theta_{ks}(X)$, are obtained from neglecting the transient terms in equations (3a,b), and directly integrating the resulting ordinary differential equations to find

$$\theta_{1s}(X) = \left(1+\frac{1+Bi_q}{Bi_q}Q\right) - Q\,X \quad (5a)$$

$$\theta_{2s}(X) = (1+Pn\,Q) - Pn\,Q\,X. \quad (5b)$$

This separation is essentially aimed at making the boundary conditions homogeneous, in order to accelerate the convergence of the eigenfunction expansions, especially in the vicinity of the boundaries, and may be accomplished by other choices of particular solutions in more involved situations [12].

The associated homogeneous problem then becomes

$$\frac{\partial\theta_{1h}(X,\tau)}{\partial\tau} = \alpha\frac{\partial^2\theta_{1h}}{\partial X^2} - \beta\frac{\partial^2\theta_{2h}}{\partial X^2} \quad (6a)$$

$$\frac{\partial\theta_{2h}(X,\tau)}{\partial\tau} = Lu\frac{\partial^2\theta_{2h}}{\partial X^2} - Lu\,Pn\frac{\partial^2\theta_{1h}}{\partial X^2} \quad (6b)$$

$$\theta_{1h}(X,0) = -\theta_{1s}(X); \quad (6c)$$

$$\theta_{2h}(X,0) = -\theta_{2s}(X), \quad 0 \leqslant X \leqslant 1 \quad (6d)$$

$$\frac{\partial\theta_{1h}(0,\tau)}{\partial X} = \frac{\partial\theta_{2h}(0,\tau)}{\partial X} = 0, \quad \tau > 0 \quad (6e,f)$$

$$\frac{\partial\theta_{1h}(1,\tau)}{\partial X} + Bi_q\theta_{1h}(1,\tau) = (1-\varepsilon)\,Ko\,Lu\,Bi_m\theta_{2h}(1,\tau), \quad \tau > 0 \quad (6g)$$

$$\frac{\partial\theta_{2h}(1,\tau)}{\partial X} + Bi_m^*\theta_{2h}(1,\tau) = -Pn\,Bi_q\theta_{1h}(1,\tau), \quad \tau > 0. \quad (6h)$$

Problem (6) above is now solved by following the formalism in the generalized integral transform technique [12–21], by taking two independent Sturm–Liouville type auxiliary problems for temperature and moisture, respectively:

$$\frac{d^2\psi_i(X)}{dX^2} + \mu_i^2\psi_i(X) = 0, \quad 0 < X < 1 \quad (7a)$$

$$\frac{d\psi_i(0)}{dX} = 0; \quad \frac{d\psi_i(1)}{dX} + Bi_q\psi_i(1) = 0 \quad (7b,c)$$

$$\frac{d^2\Gamma_i(X)}{dX^2} + \lambda_i^2\Gamma_i(X) = 0, \quad 0 < X < 1 \quad (8a)$$

$$\frac{d\Gamma_i(0)}{dX} = 0; \tag{8b}$$

$$\frac{d\Gamma_i(1)}{dX} + Bi_m^* \Gamma_i(1) = 0. \tag{8c}$$

The eigenvalue problems (7, 8) allow definition of the integral transform pairs below:

$$\bar{\theta}_{1i}(\tau) = \frac{1}{N_i^{1/2}} \int_0^1 \psi_i(X)\theta_{1h}(X,\tau)\,dX, \quad \text{transform} \tag{9a}$$

$$\theta_{1h}(X,\tau) = \sum_{i=1}^{\infty} \frac{1}{N_i^{1/2}} \psi_i(X)\bar{\theta}_{1i}(\tau), \quad \text{inverse} \tag{9b}$$

and,

$$\bar{\theta}_{2i}(\tau) = \frac{1}{M_i^{1/2}} \int_0^1 \Gamma_i(X)\theta_{2h}(X,\tau)\,dX, \quad \text{transform} \tag{10a}$$

$$\theta_{2h}(X,\tau) = \sum_{i=1}^{\infty} \frac{1}{M_i^{1/2}} \Gamma_i(X)\bar{\theta}_{2i}(\tau), \quad \text{inverse} \tag{10b}$$

where the normalization integrals are obtained from

$$N_i = \int_0^1 \psi_i^2(X)\,dX \tag{11a}$$

$$M_i = \int_0^1 \Gamma_i^2(X)\,dX. \tag{11b}$$

The auxiliary problems chosen can be readily solved to yield working expressions for eigenfunctions, eigenvalues and norms, respectively:

$$\psi_i(X) = \cos\mu_i X; \quad \Gamma_i(X) = \cos\lambda_i X \tag{12a,b}$$

$$\mu_i \tan\mu_i = Bi_q; \quad \lambda_i \tan\lambda_i = Bi_m^* \tag{12c,d}$$

$$N_i = \frac{1}{2}\left[1 + \frac{Bi_q}{\mu_i^2 + Bi_q^2}\right]; \quad M_i = \frac{1}{2}\left[1 + \frac{Bi_m^*}{\lambda_i^2 + Bi_m^{*2}}\right]. \tag{12e,f}$$

The next step is then to perform the integral transformation of the original partial differential equations, in order to reduce them into an ordinary differential system. For this purpose we operate on equation (6a) with the operator

$$\int_0^1 \frac{\psi_i}{N_i^{1/2}}\,dX$$

and on equation (6b) with

$$\int_0^1 \frac{\Gamma_i}{M_i^{1/2}}\,dX,$$

to find:

$$\frac{d\bar{\theta}_{1i}(\tau)}{d\tau} + \alpha\mu_i^2\bar{\theta}_{1i} = \frac{\alpha}{N_i^{1/2}}\left[\psi_i(1)\frac{\partial\theta_{1h}(1,\tau)}{\partial X} - \theta_{1h}(1,\tau)\frac{d\psi_i(1)}{dX}\right] - \frac{\beta}{N_i^{1/2}}\int_0^1 \psi_i \frac{\partial^2\theta_{2h}}{\partial X^2}\,dX \tag{13a}$$

$$\frac{d\bar{\theta}_{2i}(\tau)}{d\tau} + Lu\,\lambda_i^2\bar{\theta}_{2i} = \frac{Lu}{M_i^{1/2}}\left[\Gamma_i(1)\frac{\partial\theta_{2h}(1,\tau)}{\partial X} - \theta_{2h}(1,\tau)\frac{d\Gamma_i(1)}{dX}\right] - \frac{Lu\,Pn}{M_i^{1/2}}\int_0^1 \Gamma_i \frac{\partial^2\theta_{1h}}{\partial X^2}\,dX. \tag{13b}$$

The untransformed integrals represented by the coupling terms are rewritten as shown below for the first one in equation (13a):

$$\int_0^1 \psi_i \frac{\partial^2\theta_{2h}}{\partial X^2}\,dX = \int_0^1 \left[\psi_i \frac{\partial^2\theta_{2h}}{\partial X^2} - \theta_{2h}\frac{d^2\psi_i}{dX^2}\right]dX + \int_0^1 \theta_{2h}\frac{d^2\psi_i}{dX^2}\,dX. \tag{14a}$$

The first integral on the right hand side is transformed into a surface integral, in order to account for the boundaries contribution in explicit form, while the second integral is evaluated by substituting the inverse formula (10b) for $\theta_{2h}(X,\tau)$, to yield:

$$\frac{1}{N_i^{1/2}}\int_0^1 \psi_i \frac{\partial^2\theta_{2h}}{\partial X^2}\,dX = \frac{1}{N_i^{1/2}}\left[\psi_i(1)\frac{\partial\theta_{2h}(1,\tau)}{\partial X} - \theta_{2h}(1,\tau)\frac{d\psi_i(1)}{dX}\right] - \sum_{j=1}^{\infty} A_{ij}^*\bar{\theta}_{2j}(\tau) \tag{14b}$$

where,

$$A_{ij}^* = \frac{\mu_i^2}{N_i^{1/2}M_j^{1/2}}\int_0^1 \psi_i(X)\Gamma_j(X)\,dX. \tag{14c}$$

Similarly, the second untransformed integral in equation (13b) is evaluated as

$$\frac{1}{M_i^{1/2}}\int_0^1 \Gamma_i \frac{\partial^2\theta_{1h}}{\partial X^2}\,dX = \frac{1}{M_i^{1/2}}\left[\Gamma_i(1)\frac{\partial\theta_{1h}(1,\tau)}{\partial X} - \theta_{1h}(1,\tau)\frac{d\Gamma_i(1)}{dX}\right] - \sum_{j=1}^{\infty} B_{ij}^*\bar{\theta}_{1j}(\tau) \tag{14d}$$

where,

$$B_{ij}^* = \frac{\lambda_i^2}{M_i^{1/2}N_j^{1/2}}\int_0^1 \Gamma_i(X)\psi_j(X)\,dX. \tag{14e}$$

After substitution of equations (14b,d) and also making use of the boundary conditions (7c, 8c), equations (13a,b) are rewritten as:

$$\frac{d\bar{\theta}_{1i}(\tau)}{d\tau}+\alpha\mu_i^2\bar{\theta}_{1i}-\beta\sum_{j=1}^{\infty}A_{ij}^*\bar{\theta}_{2j}=\bar{g}_i(\tau) \quad (15a)$$

$$\frac{d\bar{\theta}_{2i}(\tau)}{d\tau}+Lu\lambda_i^2\bar{\theta}_{2i}-Lu\,Pn\sum_{j=1}^{\infty}B_{ij}^*\bar{\theta}_{1j}=\bar{h}_i(\tau) \quad (15b)$$

where,

$$\bar{g}_i(\tau)=\frac{\alpha\psi_i(1)}{N_i^{1/2}}\left[\frac{\partial\theta_{1h}(1,\tau)}{\partial X}+Bi_q\theta_{1h}(1,\tau)\right]-\frac{\beta\psi_i(1)}{N_i^{1/2}}\left[\frac{\partial\theta_{2h}(1,\tau)}{\partial X}+Bi_q\theta_{2h}(1,\tau)\right] \quad (15c)$$

$$\bar{h}_i(\tau)=\frac{Lu\,\Gamma_i(1)}{M_i^{1/2}}\left[\frac{\partial\theta_{2h}(1,\tau)}{\partial X}+Bi_m^*\theta_{2h}(1,\tau)\right]-\frac{Lu\,Pn\,\Gamma_i(1)}{M_i^{1/2}}\left[\frac{\partial\theta_{1h}(1,\tau)}{\partial X}+Bi_m^*\theta_{1h}(1,\tau)\right]. \quad (15d)$$

In order to complete the assembly of the O.D.E. system, the boundary quantities, $\theta_{kh}(1,\tau)$ and

$$\frac{\partial\theta_{kh}(1,\tau)}{\partial X} \quad (k=1,2),$$

must be expressed in terms of the transformed potentials, $\bar{\theta}_{ki}(\tau)$. Direct substitution of the inverse formulas (9b, 10b) at the boundaries is not, however, recommended [12, 22], since the boundary conditions for the original problem are not necessarily obeyed by the eigenfunctions. Therefore, an alternative procedure is followed here, according to the findings in refs. [12, 22], by making use of the integral balance equations to obtain rapidly converging expansions for the boundary potentials. One proceeds by integrating over the volume each of the original partial differential equations (6a,b) to find:

$$\frac{d\theta_{1,av}(\tau)}{d\tau}=\alpha\frac{\partial\theta_{1h}(1,\tau)}{\partial X}-\beta\frac{\partial\theta_{2h}(1,\tau)}{\partial X} \quad (16a)$$

$$\frac{d\theta_{2,av}(\tau)}{d\tau}=Lu\frac{\partial\theta_{2h}(1,\tau)}{\partial X}-Lu\,Pn\frac{\partial\theta_{1h}(1,\tau)}{\partial X} \quad (16b)$$

where the average potentials are defined as

$$\theta_{k,av}(\tau)=\int_0^1\theta_{kh}(X,\tau)\,dX,\quad k=1,2 \quad (17a,b)$$

and can be computed from direct substitution of the inverse formulae into the above definitions, to yield

$$\theta_{1,av}(\tau)=\sum_{j=1}^{\infty}\bar{f}_j\bar{\theta}_{1j}(\tau) \quad (17c)$$

$$\theta_{2,av}(\tau)=\sum_{j=1}^{\infty}\bar{f}_j^*\bar{\theta}_{2j}(\tau) \quad (17d)$$

where,

$$\bar{f}_j=\frac{1}{N_j^{1/2}}\int_0^1\psi_j(X)\,dX;\quad \bar{f}_j^*=\frac{1}{M_j^{1/2}}\int_0^1\Gamma_j(X)\,dX. \quad (17e,f)$$

On the other hand, the derivatives at the boundary $X=1$ can be given explicitly in terms of the potentials, through manipulation of equations (6g,h), in the form:

$$\frac{\partial\theta_{1h}(1,\tau)}{\partial X}=-Bi_q\theta_{1h}(1,\tau)+(1-\varepsilon)\,Ko\,Lu\,Bi_m\theta_{2h}(1,\tau) \quad (18a)$$

$$\frac{\partial\theta_{2h}(1,\tau)}{\partial X}=-Bi_m^*\theta_{2h}(1,\tau)-Pn\,Bi_q\theta_{1h}(1,\tau). \quad (18b)$$

By combining and solving equations (16) and (18), one obtains:

$$\theta_{1h}(1,\tau)=-\frac{1}{Bi_q}\left[\frac{d\theta_{1,av}}{d\tau}+Ko\frac{d\theta_{2,av}}{d\tau}\right] \quad (19a)$$

$$\theta_{2h}(1,\tau)=-\frac{1}{Lu\,Bi_m}\frac{d\theta_{2,av}}{d\tau} \quad (19b)$$

or,

$$\theta_{1h}(1,\tau)=-\frac{1}{Bi_q}\left[\sum_{j=1}^{\infty}\bar{f}_j\frac{d\bar{\theta}_{1j}}{d\tau}+Ko\sum_{j=1}^{\infty}\bar{f}_j^*\frac{d\bar{\theta}_{2j}}{d\tau}\right] \quad (20a)$$

$$\theta_{2h}(1,\tau)=-\frac{1}{Lu\,Bi_m}\sum_{j=1}^{\infty}\bar{f}_j^*\frac{d\bar{\theta}_{2j}}{d\tau}. \quad (20b)$$

The O.D.E. system for the transformed potentials can then be rewritten as:

$$\frac{d\bar{\theta}_{1i}(\tau)}{d\tau}+\alpha\mu_i^2\bar{\theta}_{1i}-\beta\sum_{j=1}^{\infty}A_{ij}^*\bar{\theta}_{2j}=\frac{\psi_i(1)}{N_i^{1/2}}Ko\,Lu\,[(Bi_m-\varepsilon\,Bi_q)\theta_{2h}(1,\tau)+\varepsilon\,Pn\,Bi_q\theta_{1h}(1,\tau)] \quad (21a)$$

$$\frac{d\bar{\theta}_{2i}(\tau)}{d\tau}+Lu\lambda_i^2\bar{\theta}_{2i}-Lu\,Pn\sum_{j=1}^{\infty}B_{ij}^*\bar{\theta}_{1j}=-\frac{\Gamma_i(1)}{M_i^{1/2}}Lu\,Pn\,[Bi_m^*\theta_{1h}(1,\tau)+(1-\varepsilon)Ko\,Lu\,Bi_m\theta_{2h}(1,\tau)]. \quad (21b)$$

Upon truncation to a sufficiently large finite order, N, for the desired tolerance in the converged potentials, and substitution of equations (20a,b) into the system (21) above, the following implicit O.D.E. system results:

$$\mathbf{A}\underset{\sim}{y}'+\mathbf{B}\underset{\sim}{y}=0 \quad (22a)$$

or, after inversion of the $2N\times 2N$ matrix $\mathbf{A}$

$$\underset{\sim}{y}'+\mathbf{C}\underset{\sim}{y}=0 \quad (22b)$$

where,

$$\underset{\sim}{y} = \{\bar{\theta}_{11}(\tau), \ldots, \bar{\theta}_{1N}(\tau), \bar{\theta}_{21}(\tau), \ldots, \bar{\theta}_{2N}(\tau)\}^T \quad (22c)$$

$$\mathbf{C} = \mathbf{A}^{-1}\mathbf{B}. \quad (22d)$$

The required initial conditions are represented by

$$\underset{\sim}{y}(0) = \underset{\sim}{y}_0 \quad (22e)$$

and are obtained from integral transformation of the original initial conditions (6c,d), to provide

$$\bar{F}_i = \frac{-1}{N_i^{1/2}} \int_0^1 \theta_{1s}(X)\psi_i(X)\,dX \quad (23a)$$

$$\bar{F}_i^* = \frac{-1}{M_i^{1/2}} \int_0^1 \theta_{2s}(X)\Gamma_i(X)\,dX \quad (23b)$$

and,

$$\underset{\sim}{y}_0 = \{\bar{F}_1, \ldots, \bar{F}_N, \bar{F}_1^*, \ldots, \bar{F}_N^*\}^T. \quad (23c)$$

The constant coefficients O.D.E. system formed by equations (22b,e) can be readily solved through the appropriate matrix eigensystem analysis, in the form

$$\underset{\sim}{y}(\tau) = \sum_{n=1}^{2N} c_n \, e^{-\gamma_n \tau} \underset{\sim}{\xi}^{(n)} \quad (24a)$$

where eigenvalues, γ_n, and eigenvectors, $\underset{\sim}{\xi}^{(n)}$, are obtained from the algebraic problem

$$(\mathbf{C} - \gamma\mathbf{I})\underset{\sim}{\xi} = 0 \quad (24b)$$

and the constants c_n's are evaluated so as to satisfy the initial conditions (22e), by solving a second algebraic problem

$$\sum_{n=1}^{2N} c_n \underset{\sim}{\xi}^{(n)} = \underset{\sim}{y}_0. \quad (24c)$$

The solution of problems (24b,c) is accurately and automatically accomplished through well-established algorithms readily available in scientific subroutines libraries, such as ref. [23]. Alternatively, one can directly solve the initial value problem (22b,e) through also readily accessible solvers for stiff O.D.E. systems [23].

Once the solution vector of transformed potentials, $\underset{\sim}{y}(\tau)$, is evaluated at any time τ of interest, the inverse formulae (9b) and (10b) are recalled to provide the original potentials at any position X desired. For improved convergence behavior of these eigenfunction expansions, especially in the vicinity of the boundary $X = 1$, the same procedure utilized above to compute the boundary potentials, $\theta_{kh}(1, \tau)$, can be extended to the whole medium, as suggested in refs. [12, 22]. The resulting alternative expressions for $\theta_{kh}(X, \tau)$, which can be derived in a straightforward way as now demonstrated, can then considerably enhance the convergence rates of the infinite summations.

First, both original P.D.E.'s are integrated over the region from 0 to X, providing:

$$\frac{\partial \tilde{\theta}_{1,av}(X,\tau)}{\partial \tau} = \alpha \frac{\partial \theta_{1h}(X,\tau)}{\partial X} - \beta \frac{\partial \theta_{2h}(X,\tau)}{\partial X} \quad (25a)$$

$$\frac{\partial \tilde{\theta}_{2,av}(X,\tau)}{\partial \tau} = Lu \frac{\partial \theta_{2h}(X,\tau)}{\partial X} - Lu\,Pn \frac{\partial \theta_{1h}(X,\tau)}{\partial X} \quad (25b)$$

where the quantities $\tilde{\theta}_{k,av}(X, \tau)$ are defined as

$$\tilde{\theta}_{k,av}(X,\tau) = \int_0^X \theta_{kh}(X',\tau)\,dX', \quad k = 1, 2. \quad (25c,d)$$

Equations (25a,b) are once more integrated over the region, this time from X to 1, to yield:

$$\frac{\partial \tilde{\tilde{\theta}}_{1,av}(X,\tau)}{\partial \tau} = \alpha[\theta_{1h}(1,\tau) - \theta_{1h}(X,\tau)] - \beta[\theta_{2h}(1,\tau) - \theta_{2h}(X,\tau)] \quad (26a)$$

$$\frac{\partial \tilde{\tilde{\theta}}_{2,av}(X,\tau)}{\partial \tau} = Lu[\theta_{2h}(1,\tau) - \theta_{2h}(X,\tau)] - Lu\,Pn[\theta_{1h}(1,\tau) - \theta_{1h}(X,\tau)] \quad (26b)$$

where,

$$\tilde{\tilde{\theta}}_{k,av}(X,\tau) = \int_X^1 \tilde{\theta}_{k,av}(X',\tau)\,dX', \quad k = 1, 2. \quad (26c,d)$$

The quantities above, $\tilde{\tilde{\theta}}_{k,av}$, can be directly expressed in terms of the transformed potentials, by plugging in the inverse formulae, equations (9b) and (10b), to obtain:

$$\tilde{\tilde{\theta}}_{1,av}(X,\tau) = \sum_{j=1}^{\infty} \bar{P}_j(X)\bar{\theta}_{1j}(\tau) \quad (27a)$$

$$\tilde{\tilde{\theta}}_{2,av}(X,\tau) = \sum_{j=1}^{\infty} \bar{P}_j^*(X)\bar{\theta}_{2j}(\tau) \quad (27b)$$

where,

$$\bar{P}_j(X) = \int_X^1 \int_0^{X'} \psi_j(X'')\,dX''\,dX' \quad (27c)$$

$$\bar{P}_j^*(X) = \int_X^1 \int_0^{X'} \Gamma_j(X'')\,dX''\,dX'. \quad (27d)$$

Therefore, the desired potentials, $\theta_{kh}(X, \tau)$, at any arbitrary position and time, are obtained from solution of the two algebraic equations (26a,b), and the following working expressions result:

$$\theta_{2h}(X,\tau) = \theta_{2h}(1,\tau) + \frac{Pn}{Pn\,\beta - \alpha} \sum_{j=1}^{\infty} \bar{P}_j(X) \frac{d\bar{\theta}_{1j}(\tau)}{d\tau} + \frac{\alpha}{Lu\,(Pn\,\beta - \alpha)} \sum_{j=1}^{\infty} \bar{P}_j^*(X) \frac{d\bar{\theta}_{2j}(\tau)}{d\tau} \quad (28a)$$

$$\theta_{1h}(X,\tau) = \theta_{1h}(1,\tau) + \frac{1}{\alpha}\left\{\beta[\theta_{2h}(X,\tau) - \theta_{2h}(1,\tau)] - \sum_{j=1}^{\infty} \bar{P}_j(X) \frac{d\bar{\theta}_{1j}(\tau)}{d\tau}\right\} \quad (28b)$$

where the boundary potentials are directly evaluated from the relations previously established, equations (20a,b). Similarly, if eventually needed, the derivatives within the medium can be evaluated from solution of equations (25a,b), and at the boundary from equations (18a,b).

Finally, the dimensionless temperature and moisture profiles are computed from equations (4a,b).

RESULTS AND DISCUSSION

For comparison purposes, the same numerical example considered in refs. [7, 8, 10] is here implemented. The various parameters assume the following numerical values: $Lu = 0.4$, $Pn = 0.6$, $\varepsilon = 0.2$, $Ko = 5.0$, $Bi_m = Bi_q = 2.5$, and $Q = 0.9$. The truncated system was taken with an order $N \leqslant 40$, which was more than sufficient to provide several converged significant digits on the final results at different values of dimensionless time, $\tau = 0.05$, 0.1, 0.2, 0.4, 0.8, 1.6, 3.2 and 6.4. Computation of one such complete set of results takes about 40 s on an IBM 4381 mainframe computer.

In order to illustrate the excellent convergence characteristics of the proposed eigenfunction expansions, represented by equations (28a,b), Table 1 is presented, showing both temperature and moisture distributions at different times, $\tau = 0.1$, 0.4 and 0.8 and for increasing truncation orders, $N = 5$, 10, 20 and 30. The good convergence rates are clearly noticeable; even for the smaller value of $\tau = 0.1$, the dimensionless temperature is essentially fully converged with N as low as 10, while the dimensionless moisture requires a few more terms, somewhere in between 10 and 20. For the higher values of $\tau = 0.8$, 1.6, 3.2 and 6.4, $N = 5$ is more than enough to reach the four digits accuracy shown here. The columns for $N = 30$ in Table 1, in which the results are fully converged to all four significant digits presented, provide a set of benchmark results for future reference and validation of numerical schemes.

Attention is now focused in the critical inspection of previous analytical solutions [7, 8, 10] based on application of the classical integral transform method [8], which requires in this case the analysis of a coupled eigenvalue problem that may involve a certain number of complex eigenvalues. This fact was not observed in the early contributions [7], until the work of Rossen and Hayakawa [9] appeared. Then, Lobo *et al.* [10] studied the influence of one pair of complex conjugate eigenvalues in the accuracy of temperature and moisture distributions, and concluded that especially for shorter times the expansions based on real eigenvalues could only provide completely erroneous predictions. Similar conclusions were reached in ref. [11], again with consideration of one single pair of complex eigenvalues. One advantage of the present approach is that the computation of complex eigenvalues is completely bypassed, and the computational implementation is quite straightforward, allowing for fully converged results at any prescribed accuracy. Therefore, to offer a more definitive analysis of the loss of precision in previous solutions and to access the influence of complex eigenvalues, Figs. 2(a) and (b) show, respectively,

Table 1. Convergence behavior and reference results for temperature and moisture distributions ($Lu = 0.4$, $Pn = 0.6$, $\varepsilon = 0.2$, $Ko = 5.0$, $Bi_m = 2.5$, $Bi_q = 2.5$, $Q = 0.9$)

		$\theta_1(X, \tau)$					$\theta_2(X, \tau)$		
$\tau = 0.1$									
$X \backslash N$	5	10	20	30	$X \backslash N$	5	10	20	30
0.0	0.2851	0.2850	0.2850	0.2850	0.0	0.07727	0.07725	0.07726	0.07726
0.2	0.1332	0.1330	0.1330	0.1330	0.2	0.00978	0.00987	0.00988	0.00988
0.4	0.02412	0.02408	0.02406	0.02405	0.4	0.00363	0.00368	0.00370	0.00370
0.6	−0.06171	−0.06179	−0.06179	−0.06179	0.6	0.03389	0.03431	0.03436	0.03436
0.8	−0.1204	−0.1197	−0.1196	−0.1196	0.8	0.1295	0.1304	0.1305	0.1305
1.0	−0.1112	−0.1106	−0.1106	−0.1106	1.0	0.3759	0.3769	0.3770	0.3770
$\tau = 0.4$									
$X \backslash N$	5	10	20	30	$X \backslash N$	5	10	20	30
0.0	0.4856	0.4858	0.4858	0.4859	0.0	0.2014	0.2014	0.2014	0.2014
0.2	0.3281	0.3283	0.3283	0.3283	0.2	0.1267	0.1269	0.1269	0.1269
0.4	0.2165	0.2168	0.2168	0.2168	0.4	0.1213	0.1214	0.1215	0.1215
0.6	0.1536	0.1539	0.1539	0.1539	0.6	0.1903	0.1906	0.1906	0.1906
0.8	0.1416	0.1420	0.1420	0.1420	0.8	0.3388	0.3389	0.3389	0.3389
1.0	0.1791	0.1794	0.1794	0.1794	1.0	0.5616	0.5618	0.5618	0.5618
$\tau = 0.8$									
$X \backslash N$	5	10	20	30	$X \backslash N$	5	10	20	30
0.0	0.7769	0.7771	0.7771	0.7771	0.0	0.3709	0.3710	0.3710	0.3710
0.2	0.6193	0.6196	0.6196	0.6196	0.2	0.2980	0.2982	0.2982	0.2982
0.4	0.5061	0.5063	0.5064	0.5064	0.4	0.2945	0.2946	0.2946	0.2946
0.6	0.4354	0.4356	0.4356	0.4356	0.6	0.3558	0.3569	0.3569	0.3569
0.8	0.4035	0.4037	0.4037	0.4037	0.8	0.4781	0.4780	0.4780	0.4780
1.0	0.4043	0.4044	0.4044	0.4044	1.0	0.6457	0.6458	0.6458	0.6458

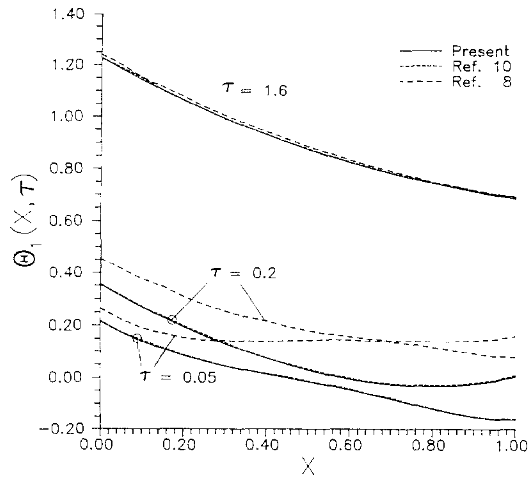

Fig. 2(a). Comparison of dimensionless temperature distributions with previously reported analytical solutions ($Lu = 0.4$, $Pn = 0.6$, $\varepsilon = 0.2$, $Ko = 5.0$, $Bi_m = 2.5$, $Bi_q = 2.5$, $Q = 0.9$).

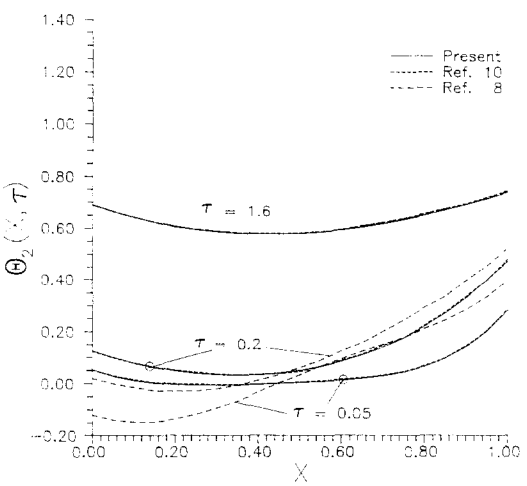

Fig. 2(b). Comparison of dimensionless moisture distributions with previously reported analytical solutions ($Lu = 0.4$, $Pn = 0.6$, $\varepsilon = 0.2$, $Ko = 5.0$, $Bi_m = 2.5$, $Bi_q = 2.5$, $Q = 0.9$).

dimensionless temperature and moisture distributions for different times, $\tau = 0.05$, 0.2, and 1.6. Besides the results from the approach here proposed, the results from ref. [7], which does not account for complex eigenvalues, and from ref. [10], which includes one pair of complex conjugate roots, are also plotted. For larger values of τ all the solutions provide essentially the same results, but as τ is decreased the effect of the complex eigenvalues become more and more relevant. For $\tau = 0.2$ the results from ref. [7] are already meaningless, while for $\tau = 0.05$, the inclusion of one pair of complex roots [10], to the present graph scale, seems enough to provide a good agreement with the fully converged results. Additional complex roots might be required for smaller values of τ. The same behavior is observed in both temperature and moisture distributions.

The present analytical approach has been successfully applied to a classical problem in coupled heat and mass transfer, represented by Luikov's equations for drying in capillary porous media. It offers an interesting alternative in practical applications as well as in extensions to more realistic situations of nonlinear problems and multidimensional geometries [24, 25], by combining previous recent developments on the generalized integral transform technique [12–21]. For instance, variable transport coefficients can be accounted for by incorporating the extension to nonlinear problems discussed in refs. [12–14]. Quite recently, the present approach was utilized in the solution of drying problems involving radiative boundary conditions [26], which illustrates the potential of the method to handle more involved applications.

Acknowledgements—One of the authors (J.W.R.) wishes to acknowledge the financial support provided by the Universidade Federal do Ceará (UFC) and CAPES/MEC, both from Brazil. The kind hospitality of the Mechanical Engineering Department at EE/COPPE/UFRJ and financial support from CNPq, during the visit of Professor M. D. Mikhailov, is also deeply acknowledged.

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